

Coarse embeddings into $c_0(\Gamma)$

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Abstract

Let λ be a large enough cardinal number (assuming GCH it suffices to let $\lambda = \aleph_\omega$). If X is a Banach space with $\text{dens}(X) \geq \lambda$, which admits a coarse (or uniform) embedding into any $c_0(\Gamma)$, then X fails to have nontrivial cotype, i.e. X contains ℓ_∞^n C -uniformly for every $C > 1$. In the special case when X has a symmetric basis, we may even conclude that it is linearly isomorphic with $c_0(\text{dens} X)$.

1 Introduction

The classical result of Aharoni states that every separable metric space (in particular every separable Banach space) can be bi-Lipschitz embedded (the definition is given below) into c_0 .

The natural problem of embeddings of metric spaces into $c_0(\Gamma)$, for an arbitrary set Γ , has been treated by several authors, in particular Pelant and Swift.

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The characterizations that they obtained, and which play a crucial role in our argument, are described below.

Our main interest, motivated by some problems posed in [3], lies in the case of embeddings of Banach spaces into $c_0(\Gamma)$.

We now state the main results of this paper. We first define the following cardinal numbers inductively. We put $\lambda_0 = \omega_0$, and, assuming for $n \in \mathbb{N}_0$, λ_n has been defined, we put $\lambda_{n+1} = 2^{\lambda_n}$. Then we let

$$\lambda = \lim_{n \rightarrow \infty} \lambda_n. \quad (1)$$

It is clear that assuming the generalized continuum hypothesis (GCH) $\lambda = \aleph_\omega$.

Theorem A. If X is a Banach space with density $\text{dens}(X) \geq \lambda$, which admits a coarse (or uniform) embedding into any $c_0(\Gamma)$, then X fails to have nontrivial cotype, i.e. X contains ℓ_∞^n C -uniformly for some $C > 1$ (equivalently, every $C > 1$).

Our method of proof gives a much stronger result for Banach spaces with a symmetric basis. Namely, under the assumptions of Theorem A, such spaces are linearly isomorphic with $c_0(\Gamma)$ (Theorem (4.2)).

Theorem A will follow from the following combinatorial result which is of independent interest.

Theorem B. Assume that Λ is a set whose cardinality is at least λ , $n \in \mathbb{N}$, and $\sigma : [\Lambda]^n \rightarrow \mathcal{C}$ is a map into an arbitrary set \mathcal{C} . Then (at least) one of the following conditions holds:

1. There is a sequence $(F_j)_{j=1}^\infty$ of pairwise disjoint elements of $[\Lambda]^n$, so that $\sigma(F_i) = \sigma(F_j)$, for all $i, j \in \mathbb{N}$.
2. There is an $F \in [\Lambda]^{n-1}$ so that $\sigma(\{F \cup \{\gamma\} : \gamma \in \Lambda\})$ is infinite.

The above Theorem B was previously deduced in [6] from a combinatorial result of Baumgartner, provided Λ is a weakly compact cardinal number (whose existence is not provable in ZFC, as it is inaccessible [5] p. 325, p. 52). The authors in [6] pose a question whether assuming that Γ is uncountable is sufficient in Theorem B.

Theorem B is used in order to obtain a scattered compact set K of height ω_0 , such that $C(K)$ does not uniformly embed into $c_0(\Gamma)$. It is easy to check that our version of Theorem B implies a ZFC example of such a $C(K)$ space. It is further shown in [6] that the space $C[0, \omega_1]$ does not uniformly embed into any $c_0(\Gamma)$.

Let us point out that a special case of Theorem A was obtained by Pelant and Rodl [8], namely it was shown there that $\ell_p(\lambda)$, $1 \leq p < \infty$, spaces (which are well known to have nontrivial cotype) do not uniformly embed into any $c_0(\Gamma)$.

The paper is organized as follows. In Section 2 we recall Pelant's [7, 6] and Swift's [10] conditions for Lipschitz, uniform, and coarse embeddability into

$c_0(\Gamma)$. In Section 3 we provide a proof for Theorem B. Finally, in Section 4, we provide a proof of Theorem A as well as the symmetric version of the result.

All set theoretic concepts and results used in our note can be found in [5], whereas for facts concerning nonseparable Banach spaces [4] can be consulted.

2 Pelant's and Swift's criteria for Lipschitz, uniform, and coarse embeddability into $c_0(\Gamma)$

In this section we recall some of the notions and results by Pelant [7, 6] and Swift [10] about embeddings into $c_0(\Gamma)$.

For a metric space (M, d) a *cover* is a set \mathcal{U} of subsets of M such that $M = \bigcup_{U \in \mathcal{U}} U$. A cover \mathcal{U} of M is called *uniform* if there is an $r > 0$ so that for all $x \in M$ there is a $U \in \mathcal{U}$, so that $B_r(x) = \{x' \in M : d(x', x) < r\} \subset U$. It is called *uniformly bounded* if the diameters of the $U \in \mathcal{U}$ are uniformly bounded, and it is called *point finite* if every $x \in M$ lies in only finitely many $U \in \mathcal{U}$. A cover \mathcal{V} of M is a *refinement* of a cover \mathcal{U} , if for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$, for which $V \subset U$.

Definition 2.1. [6] A metric space (M, d) is said to have the *Uniform Stone Property* if every uniform cover \mathcal{U} of M has a point finite uniform refinement.

Definition 2.2. [10] A metric space (M, d) is said to have the *Coarse Stone Property* if every bounded cover is the refinement of a point finite uniformly bounded cover.

Definition 2.3. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. For a map $f : M_1 \rightarrow M_2$ we define the *modulus of uniform continuity* $w_f : [0, \infty) \rightarrow [0, \infty]$, and the *modulus of expansion* $\rho : [0, \infty) \rightarrow [0, \infty]$ as follows

$$w_f(t) = \sup \{d_2(f(x), f(y)) : x, y \in M_1, d_1(x, y) \leq t\} \text{ and} \\ \rho_f(t) = \inf \{d_2(f(x), f(y)) : x, y \in M_1, d_1(x, y) \geq t\}.$$

The map f is called *uniform continuous* if $\lim_{t \rightarrow 0} w_f(t) = 0$, and it is called a *uniform embedding* if moreover $\rho_f(t) > 0$ for every $t > 0$. It is called *coarse* if $w_f(t) < \infty$, for all $0 < t < \infty$ and it is called a *coarse embedding*, if $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$. The map f is called *Lipschitz continuous* if

$$\text{Lip}(f) = \sup_{x \neq y} \frac{d_2(f(x), f(y))}{d_1(x, y)} < \infty,$$

and a bi-Lipschitz embedding, if f is injective and $\text{Lip}(f^{-1})$ is also finite.

The following result recalls results from [6] (for (i) \iff (ii)) and [10] (for (ii) \iff (iii) \iff (iv) \iff (v)).

Theorem 2.4. For a Banach space X the following properties are equivalent.

- (i) X has the uniform Stone Property.

- (ii) X is uniformly embeddable into $c_0(\Gamma)$, for some set Γ .
- (iii) X has the coarse Stone Property.
- (iv) X is coarsely embeddable into $c_0(\Gamma)$, for some set Γ .
- (v) X is bi-Lipschitzly embeddable into $c_0(\Gamma)$, for some set Γ .

It is easy to see, and was noted in [6, 10], that the uniform Stone property and the coarse Stone property are inherited by subspaces. The equivalence (i) \iff (ii) was used in [6] to show that $C[0, \omega_1]$ does not uniformly embed in any $c_0(\Gamma)$. It was also used to prove that certain other $C(K)$ -spaces do not uniformly embed into $c_0(\Gamma)$: Let Λ be any set and denote for $n \in \mathbb{N}$ by $[\Lambda]^{\leq n}$ and $[\Lambda]^n$ the subsets of Λ which have cardinality at most n and exactly n , respectively. Endow $[\Lambda]^{\leq n}$ with the restriction of the product topology on $\{0, 1\}^\Lambda$ (by identifying each set with its characteristic function). Then define K_Λ to be the one-point Alexandroff compactification of the topological sum of the spaces $[\Lambda]^{\leq n}$, $n \in \mathbb{N}$. It was shown in [6] that if Λ satisfies Theorem B then $C(K_\Lambda)$ is not uniformly Stone and thus does not embed uniformly into any $c_0(\Gamma)$.

3 A combinatorial argument

We start by introducing property $P(\alpha)$ for a cardinal α as follows.

For every $n \in \mathbb{N}$ and any map $\sigma : [\alpha]^n \rightarrow \mathcal{C}$, \mathcal{C} being an arbitrary set, $(P(\alpha))$ (at least) one of the following two conditions hold:

There is a sequence (F_j) of pairwise disjoint elements of $[\lambda]^n$, with $\sigma(F_i) = \sigma(F_j)$,
(2)

for any $i, j \in \mathbb{N}$.

There is an $F \in [\lambda]^{n-1}$, so that $\sigma(\{F \cup \{\gamma\} : \gamma \in \lambda \setminus F\})$ is infinite. (3)

As remarked in Section 2, if κ is an uncountable weakly compact cardinal number, then $P(\kappa)$ holds. But the existence of weakly compact cardinal numbers requires further set theoretic axioms, beyond ZFC [5]. In [6, Question 3] the authors ask if $P(\omega_1)$ is true.

Theorem 3.1. *For λ defined by (1), $P(\lambda)$ holds.*

For our proof of Theorem 3.1 it will be more convenient to reformulate it into a statement about n -tuples, instead of sets of cardinality n . We will first introduce some notation.

Let $n \in \mathbb{N}$ and $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be sets of infinite cardinality, and put $\Gamma = \prod_{i=1}^n \Gamma_i$. For $a \in \Gamma$ and $1 \leq i \leq n$ we denote the i -th coordinate of a by $a(i)$. We say that two points a and b in Γ are *diagonal*, if $a(i) \neq b(i)$, for all $i \in \{1, 2, \dots, n\}$.

Let $a \in \Gamma_i$ for $i \in \mathbb{N}$. For $i \in \{1, 2, \dots, n\}$ we call the set

$$H(a, i) = \{(b_1, b_2, \dots, b_{i-1}, a(i), b_{i+1}, \dots, \alpha_n) : b_j \in \Gamma_j, \text{ for } j \in \{1, 2, \dots, n\} \setminus \{i\}\},$$

the *Hyperplane through the point a orthogonal to i* . We call the set

$$L(a, i) = \{(a(1), \dots, a(i-1), b_i, a(i+1), \dots, a(n)) : b_i \in \Gamma_i\},$$

the *Line through the point a in direction of i* .

For a cardinal number β , we define recursively the following sequence of cardinal numbers $(\exp_+(\beta, n) : n \in \mathbb{N}_0)$: $\exp_+(\beta, 0) = \beta$, and, assuming $\exp_+(\beta, n)$ has been defined for some $n \in \mathbb{N}_0$, we put

$$\exp_+(\beta, n+1) = (2^{\exp_+(\beta, n)+})^+.$$

Here γ^+ denotes the *successor cardinal*, for a cardinal γ , i.e., the smallest cardinal number γ' with $\gamma' > \gamma$. Note that since $\exp_+(\gamma, 1) \leq 2^{2^{2^\gamma}}$, it follows for the above defined cardinal number λ , that

$$\lambda = \lim_{n \rightarrow \infty} \exp_+(\omega_0, n).$$

Secondly, successor cardinals are regular [5], and thus every set of cardinality γ , with γ being a successor cardinal, can be partitioned for $n \in \mathbb{N}$ into n disjoint sets $\Gamma_1, \Gamma_2, \dots, \Gamma_n$, all of them having also cardinality γ , and the map $\Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n \rightarrow [\bigcup_{i=1}^n \Gamma_i]^n$, $(a_1, a_2, \dots, a_n) \mapsto \{a_1, a_2, \dots, a_n\}$, is injective. We therefore deduce that the following statement implies Theorem 3.1.

Theorem 3.2. *Let $n \in \mathbb{N}$ and assume that the sets $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ have cardinality at least $\exp_+(\omega_1, n^2)$. For any function*

$$\sigma : \Gamma := \prod_{i=1}^n \Gamma_i \rightarrow \mathcal{C},$$

where \mathcal{C} is an arbitrary set, at least one of the following two conditions hold

There is a sequence $(a^{(j)})_{j=1}^\infty$, of pairwise diagonal elements in Γ , so that (4)

$$\sigma(a^{(i)}) = \sigma(a^{(j)}), \text{ for any } i, j \in \mathbb{N}.$$

There is a line $L \subset \Gamma$, for which $\sigma(L)$ is infinite. (5)

We will make the following observation before proving Theorem 3.2.

Lemma 3.3. *Let $n \in \mathbb{N}$ and $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be non empty sets. Let*

$$\sigma : \Gamma := \prod_{i=1}^n \Gamma_i \rightarrow \mathcal{C},$$

be a function that fails both conditions (4) and (5).

Then there is a set $\tilde{\mathcal{C}}$ and a function

$$\tilde{\sigma} : \Gamma := \prod_{i=1}^n \Gamma_i \rightarrow \tilde{\mathcal{C}},$$

that fails both (4) and (5) and moreover has the property that

$$\text{for every } c \in \mathcal{C} \text{ there is a hyperplane } H_c \subset \Gamma \text{ so that } \{b \in \Gamma : \tilde{\sigma}(b) = c\} \subset H_c. \quad (6)$$

Proof. We may assume without loss of generality that σ is surjective. Since (4) is not satisfied for each $c \in \mathcal{C}$ there exists an $m(c) \in \mathbb{N}$ and a (finite) sequence $(a^{(c,j)})_{j=1}^{m(c)} \subset \sigma^{-1}(\{c\})$, which is pairwise diagonal, and maximal, with this property. Hence

$$\sigma^{-1}(\{c\}) \subset \bigcup_{j=1}^{m(c)} \bigcup_{i=1}^n H(a^{(c,j)}, i).$$

Indeed, from the maximality of $(a^{(c,j)})_{j=1}^{m(c)} \subset \sigma^{-1}(\{c\})$, it follows that each $b \in \sigma^{-1}(\{b\})$ must have at least one coordinate in common with at least one element of $(a^{(c,j)})_{j=1}^{m(c)} \subset \sigma^{-1}(\{c\})$.

We define

$$\tilde{\mathcal{C}} = \bigcup_{c \in \mathcal{C}} \{1, 2, \dots, m(c)\} \times \{1, 2, \dots, n\} \times \{c\},$$

and

$$\tilde{\sigma} : \Gamma \rightarrow \tilde{\mathcal{C}}, \quad b \mapsto (c, j, i), \text{ where}$$

$$c = \sigma(b), \quad j = \min \left\{ j' : b \in \bigcup_{i'=1}^n H(a^{(c,j')}, i') \right\}, \text{ and } i = \min \{ i' : b \in H(a^{(c,j)}, i) \}.$$

It is clear that $\tilde{\sigma}$ satisfies (6). Since for every $c \in \mathcal{C}$,

$$\{b \in \Gamma : \sigma(b) = c\} = \bigcup_{j=1}^{m(c)} \bigcup_{i=1}^n \{b \in \Gamma : \tilde{\sigma}(b) = (c, j, i)\},$$

$\tilde{\sigma}$ does not satisfy (4). In order to verify that (5) is not satisfied, assume $L \subset \Gamma$ is a line, and let $\{c_1, c_2, \dots, c_p\}$ be the image of L under σ . By construction,

$$\tilde{\sigma}(L) \subset \{(j, i, c_k), k \leq p, j \leq m(c_k), 1 \leq n\},$$

which is also finite. □

Proof of Theorem 3.2. We assume that $\sigma : \Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n \rightarrow \mathcal{C}$ is a map which fails both (4) and (5). By Lemma 3.3 we may also assume that σ satisfies (6). For each $a \in \Gamma$ we fix an $i(a) \in \{1, 2, \dots, n\}$ so that $\sigma^{-1}(\{\sigma(a)\}) \subset H(a, i(a))$. It is important to note that, since (5) is not satisfied, it follows that

each line L , whose direction is some $j \in \{1, 2, \dots, n\}$, can only have finitely many elements b for which $i(b) = j$. Indeed, if $i(b) = j$ then b is uniquely determined by the value $\sigma(b)$. To continue with the the proof the following *Reduction Lemma* will be essential.

Lemma 3.4. *Let β be an uncountable regular cardinal. Assume that $\tilde{\Gamma}_1 \subset \Gamma_1$, $\tilde{\Gamma}_2 \subset \Gamma_2, \dots, \tilde{\Gamma}_n \subset \Gamma_n$ are such that $|\tilde{\Gamma}_i| \geq \exp_+(\beta, n)$, for all $i \in \{1, \dots, n\}$.*

Then, for any $i \in \{1, 2, \dots, n\}$ there are a number $K_i \in \mathbb{N}$, and subsets $\Gamma'_1 \subset \tilde{\Gamma}_1, \Gamma'_2 \subset \tilde{\Gamma}_2, \dots, \Gamma'_n \subset \tilde{\Gamma}_n$, with $|\Gamma'_j| \geq \beta$, so that

$$\begin{aligned} \forall (a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in \prod_{j=1, j \neq i}^n \Gamma'_j \\ |\{a \in \Gamma'_i : i(a_1, a_2, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) = i\}| \leq K_i. \end{aligned} \quad (7)$$

Proof. We assume without loss of generality that $i = n$. Abbreviate $\beta_j = \exp_+(\beta, j)$, for $j = 1, 2, \dots, n$. We choose subsets $\tilde{\Gamma}_j^{(0)} \subset \tilde{\Gamma}_j$, for which $|\tilde{\Gamma}_j^{(0)}| = \beta_{n+1-j}$.

Since the β_j 's are regular, it follows for each $j = 1, 2, \dots, n-1$ that

$$\begin{aligned} |\tilde{\Gamma}_j^{(0)}| &= \beta_{n+1-j} \\ &> 2^{\beta_{n-j}} \\ &= 2^{|\tilde{\Gamma}_{j+1}^{(0)} \times \tilde{\Gamma}_{j+1}^{(0)} \times \dots \times \tilde{\Gamma}_n^{(0)}|} \\ &= |\{f : \tilde{\Gamma}_{j+1}^{(0)} \times \tilde{\Gamma}_{j+1}^{(0)} \times \dots \times \tilde{\Gamma}_n^{(0)} \rightarrow \mathbb{N}\}|. \end{aligned}$$

Abbreviate for $i = 1, \dots, n$.

$$\mathcal{F}_j = \{f : \tilde{\Gamma}_j^{(0)} \times \tilde{\Gamma}_{j+1}^{(0)} \times \dots \times \tilde{\Gamma}_n^{(0)} \rightarrow \mathbb{N}\}$$

and consider the function

$$\phi_1 : \prod_{j=1}^{n-1} \tilde{\Gamma}_j^{(0)} \rightarrow \mathbb{N}, \quad (a_1, a_2, \dots, a_{n-1}) \mapsto |\{a \in \tilde{\Gamma}_n^{(0)} : i(a_1, a_2, \dots, a_{n-1}, a) = n\}|.$$

For fixed $a_1 \in \Gamma_1^{(0)}$, $\phi_1(a_1, \cdot) \in \mathcal{F}_2$, and the cardinality of \mathcal{F}_2 is by the above estimates smaller than the cardinality of $\tilde{\Gamma}_1^{(0)}$, which is regular. Therefore we can find a function $\Phi_2 \in \mathcal{F}_2$ and a subset $\Gamma'_1 \subset \tilde{\Gamma}_1^{(0)}$ of cardinality β_n so that $\phi_1(a_1, \cdot) = \Phi_2$ for all $a_1 \in \Gamma'_1$. We continue the process and find $\Gamma'_j \subset \tilde{\Gamma}_j^{(0)}$, for $j = 1, 2, \dots, n-2$ of cardinality β_{n+1-j} and functions $\phi_j \in \mathcal{F}_j$, for $j = 1, 2, \dots, n-1$, so that for all $(a_1, a_2, \dots, a_{n-2}) \in \prod_{j=1}^{n-2} \Gamma'_j$ and $a_{n-1} \in \tilde{\Gamma}_{n-1}^{(0)}$, we have

$$\phi_1(a_1, a_2, \dots, a_{n-1}) = \phi_2(a_2, \dots, a_{n-1}) = \dots = \phi_{n-1}(a_{n-1}). \quad (8)$$

Then, since ϕ_{n-1} is \mathbb{N} valued, we can finally choose an $K_n \in \mathbb{N}$ and a subset Γ'_{n-1} , of cardinality at least β , so that $\phi_{n-1}(a_1) \leq K_n$, which finishes our argument. \square

Continuation of the proof of Theorem 3.2. We apply Lemma 3.4 successively to all $i \in \{1, 2, \dots, n\}$, and the cardinals $\beta^{(i)} = \exp_+(\omega_1, n(n-i))$. We obtain numbers K_1, K_2, \dots, K_n in \mathbb{N} and infinite sets $\Lambda_j \subset \Gamma_i$, for $j \leq n$, so that for all $i = 1, 2, \dots, n$ and all $a \in \prod_{j=1}^n \Lambda_j$

$$|\{a \in \Lambda_i : i(a_1, a_2, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) = i\}| \leq K_i.$$

In order to deduce a contradiction choose for each $j = 1, \dots, n$ a subset A_j of Λ_j of cardinality $l_j = (n+1)K_j$. Then it follows that

$$\begin{aligned} \prod_{j=1}^n l_j &= \left| \prod_{j=1}^n A_j \right| \\ &= \sum_{i=1}^n \sum_{a \in \prod_{j=1, j \neq i}^n A_j} |\{a \in A_i : i(a_1, a_2, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) = i\}| \\ &\leq \sum_{i=1}^n K_i \prod_{j=1, j \neq i}^n l_j \leq \frac{n}{n+1} \prod_{j=1}^n l_j \end{aligned}$$

which is a contradiction and finishes the proof of the Theorem. \square

We can now state the ZFC version of Theorem 4.1. in [6], in which it was shown that for weakly compact cardinalities κ_0 the space $C(K_{\kappa_0})$, where K_{κ_0} was defined at the end of Section 2, cannot be uniformly (or coarsely) embedded into any $c_0(\Gamma)$, where Γ has any cardinality. Since the only property of κ_0 , which is needed in [6], is the fact that $(P(\kappa_0))$ holds, we deduce

Corollary 3.5. *$C(K_\lambda)$ does not coarsely (or uniformly) embed into $c_0(\Gamma)$, for any cardinality Γ .*

4 Proof of Theorem A

In this section we use our combinatorial Theorem B from Section 3 to show Theorem A.

Recall that a long Schauder basis of a Banach space X is a transfinite sequence $\{e_\gamma\}_{\gamma=0}^\Gamma$ such that for every $x \in X$ there exists a unique transfinite sequence of scalars $\{a_\gamma\}_{\gamma=0}^\Gamma$ such that $x = \sum_{\gamma=0}^\Gamma a_\gamma e_\gamma$. Similarly, a long Schauder basic sequence in a Banach space X is a transfinite sequence $\{e_\gamma\}_{\gamma=0}^\Gamma$ which is a long Schauder basis of its closed linear span. Recall that the w^* -dens(X^*) is the smallest cardinal such that there exists a w^* -dense subset of X^* . Analogously to the classical Mazur construction of a Schauder basic sequence in a separable Banach space we have the following result, proved e.g. in [4, p.135] (the fact that the basis is normalized, i.e. $\|e_\gamma\| = 1$, is not a part of the statement in [4], but it is easy to get it by normalizing the existing basis).

Theorem 4.1. *Let X be a Banach space with $\Gamma = w^* - \text{dens} X^* > \omega_0$. Then X contains a monotone normalized long Schauder basic sequence of length Γ .*

Proof of Theorem A. Using the Hahn-Banach theorem it is easy to see that $w^* - \text{dens} X^* \leq \text{dens} X$. On the other hand, since every $x \in X$ is uniquely determined by its values on a w^* -dense subset of X^* , it is clear that

$$\text{dens} X \leq \text{card} X \leq 2^{w^* - \text{dens} X^*}$$

It follows that for λ defined in (1) we get that $\lambda = w^* - \text{dens} X^*$ holds if and only if $\lambda = \text{dens} X$. In order to prove Theorem A we may assume without loss of generality that X has a long normalized and monotone Schauder basis $(e_\mu)_{\mu < \lambda}$, of length λ , i.e. $\Gamma = \lambda$.

Set

$$D_n = \{F \subset \lambda : |F| = n\}, \quad n \in \mathbb{N}$$

Suppose that $F = \{\gamma_1, \dots, \gamma_n\}$ where $\gamma_1 < \dots < \gamma_n$ are elements of $[0, \lambda)$ arranged in an increasing order. Consider the corresponding finite set $M_F = \{\sum_{i=1}^n \varepsilon_i e_{\gamma_i} : \varepsilon_i \in \{-1, 1\}\}$, containing 2^n distinct vectors of X , and put a linear order \prec on this set according to the arrangement of the signs ε_i , setting

$$\sum_{i=1}^n \varepsilon_i e_{\gamma_i} \prec \sum_{i=1}^n \tilde{\varepsilon}_i e_{\gamma_i}$$

if and only if for the minimal i , such that $\varepsilon_i \neq \tilde{\varepsilon}_i$, it holds $\varepsilon_i < \tilde{\varepsilon}_i$. In order to prove Theorem A it suffices to show that if $M = \cup_{F \in D_n, n \in \mathbb{N}} M_F \subset X$ has the coarse Stone property then X fails to have nontrivial cotype. To this end, starting with $\mathcal{U} = \{B_2(x) : x \in M\}$ we find a uniform bounded cover \mathcal{V} , which is point finite and so that \mathcal{U} refines \mathcal{V} , i.e., for all $x \in M$ there is a $V_x \in \mathcal{V}$ with $B_2(x) \subset V_x$. Let $r > 0$ be such that each $V \in \mathcal{V}$ is a subset of a ball of radius r .

Let \mathcal{C} be the set consisting of all finite tuples (V^1, \dots, V^m) , where $V^j \in \mathcal{V}$. We now define the function $\sigma : M \rightarrow \mathcal{C}$ as follows. If $F \in D_n$, $F = \{\gamma_1, \dots, \gamma_n\}$ where $\gamma_1 < \dots < \gamma_n$, we let

$$\sigma(F) = (V_{y_1}, \dots, V_{y_{2^n}}), \quad (9)$$

where $y_1 \prec \dots \prec y_{2^n}$ are the elements of M_F arranged in the increasing order. Applying Theorem B to the function σ , for a fixed $n \in \mathbb{N}$, yields one of two possibilities. Either there is an $F = \{\gamma_1, \dots, \gamma_{n-1}\}$, where $\gamma_1 < \dots < \gamma_{n-1}$, so that $\sigma(\{F \cup \{\tau\} : \tau \in \lambda \setminus F\})$ is infinite. In this case, pick an infinite sequence of distinct $\{\tau_j\}_{j=1}^\infty$ witnessing the desired property. By passing to a subsequence, we may assume without loss of generality that either there exists $k, 1 \leq k \leq n-1$, so that for all $j \in \mathbb{N}$, $\gamma_k < \tau_j < \gamma_{k+1}$, or $\tau_j < \gamma_1$ for all $j \in \mathbb{N}$, or $\gamma_{n-1} < \tau_j$ for all $j \in \mathbb{N}$. For simplicity of notation, assume the last case, i.e. $\gamma_1 < \dots < \gamma_{n-1} < \tau_j$ holds for all $j \in \mathbb{N}$. Denoting $F^j = \{\gamma_1, \dots, \gamma_{n-1}, \tau_j\}$, we conclude that there exists a fixed selection of signs $\varepsilon_1, \dots, \varepsilon_n$ such that the set

$$B = \left\{ V_y : y = \sum_{i=1}^{n-1} \varepsilon_i e_{\gamma_i} + \varepsilon_n \tau_j, j \in \mathbb{N} \right\}$$

is infinite. Indeed, otherwise the set of values $\{\sigma(\{\gamma_1, \dots, \gamma_{n-1}, \tau_j\}), j \in \mathbb{N}\}$, which are determined by the definition (9), would have only a finite set of options for each coordinate, and would therefore have to be finite. This is a contradiction with the point finiteness of the system \mathcal{V} , because

$$\sum_{i=1}^{n-1} \varepsilon_i e_{\gamma_i} \in V_y, \text{ for all } V_y \in B.$$

It remains to consider the other case when there is a sequence (F_j) of pairwise disjoint elements of $[\lambda]^n$, with $\sigma(F_i) = \sigma(F_j)$, for any $i, j \in \mathbb{N}$. In fact, it suffices to choose just a pair of such disjoint elements (written in an increasing order of ordinals) $F = \{\gamma_1, \dots, \gamma_n\}$, $G = \{\beta_1, \dots, \beta_n\}$, such that $\sigma(F) = \sigma(G)$. This means, in particular, that for every fixed selection of signs $\varepsilon_1, \dots, \varepsilon_n$,

$$V_{\sum_{i=1}^n \varepsilon_i e_{\gamma_i}} = V_{\sum_{i=1}^n \varepsilon_i e_{\beta_i}}$$

By our assumption, the elements of \mathcal{V} are contained in a ball of radius r , hence

$$\left\| \sum_{i=1}^n \varepsilon_i e_{\gamma_i} - \sum_{i=1}^n \varepsilon_i e_{\beta_i} \right\| \leq 2r \quad (10)$$

holds for any selection of signs $\varepsilon_1, \dots, \varepsilon_n$. Let $u_j = e_{\gamma_j} - e_{\beta_j}$, $j \in \{1, \dots, n\}$. Because $\{e_\gamma\}$ is a monotone normalized long Schauder basis, we have the trivial estimate $1 \leq \|u_j\| \leq 2$. The equation (10) means that

$$1 \leq \left\| \sum_{i=1}^n \varepsilon_i u_i \right\| \leq 2r \quad (11)$$

holds for any selection of signs $\varepsilon_1, \dots, \varepsilon_n$. Since norm functions are convex, this means that for the unite vector ball B_E of $E = \text{span}(u_i : i \leq n)$ it follows that

$$\left\{ \sum_{j=1}^n a_j u_j : |a_j| \leq \frac{1}{2r} \right\} \subset B_E \subset \left\{ \sum_{j=1}^n a_j u_j : |a_j| \leq 2 \right\},$$

which means that $(u_j)_{j=1}^n$ is $4r$ -equivalent to the unit vector basis of ℓ_∞^n . \square

In fact, our proof gives a much stronger condition than just failing cotype, because our copies of ℓ_∞^k are formed by vectors of the type $e_\alpha - e_\beta$. This fact can be used to obtain much stronger structural results for spaces with special bases. Recall that a long Schauder basis $\{e_\gamma\}_{\gamma=1}^\Lambda$ is said to be symmetric if

$$\left\| \sum_{i=1}^n a_i e_{\gamma_i} \right\| = \left\| \sum_{i=1}^n a_i e_{\beta_i} \right\|$$

for any selection of $a_i \in \mathbb{R}$, and any pair of sets $\{\gamma_i\}_{i=1}^n \subset [1, \Lambda)$, $\{\beta_i\}_{i=1}^n \subset [1, \Lambda)$. It is well-known (c.f. [9, Prop. II.22.2]), that each symmetric basis is automatically unconditional, i.e. there exists $K > 0$ such that

$$\frac{1}{K} \left\| \sum_{i=1}^n |a_i| e_{\gamma_i} \right\| \leq \left\| \sum_{i=1}^n a_i e_{\gamma_i} \right\| \leq K \left\| \sum_{i=1}^n |a_i| e_{\gamma_i} \right\|.$$

In particular,

$$\frac{1}{K} \left\| \sum_{i \in A} a_i e_{\gamma_i} \right\| \leq \left\| \sum_{i \in B} a_i e_{\gamma_i} \right\|$$

whenever $A \subset B$.

Theorem 4.2. *Let X be a Banach space of density $\Lambda \geq \lambda$, with a symmetric basis $\{e_\gamma\}_{\gamma=1}^\Lambda$, which coarsely (or uniformly) embeds into some $c_0(\Gamma)$.*

Then X is linearly isomorphic with $c_0(\Lambda)$.

Proof. By the proof of the above results, if X embeds into $c_0(\Gamma)$, there exists an $C > 0$, such that for each $k \in \mathbb{N}$ there are some vectors $\{v_i\}_{i=1}^k$ of the form $v_i = e_{\gamma_i} - e_{\beta_i}$ satisfying the conditions

$$\frac{1}{2} \max_j |a_j| \leq \left\| \sum_{i=1}^k a_i v_i \right\| \leq C \max_j |a_j|. \quad (12)$$

Using the fact that the basis $\{e_\gamma\}$ is unconditional, (and symmetric) we obtain by an easy manipulation that there exist some constants $A, B > 0$ such that

$$A \left\| \sum_{i=1}^k a_i e_{\gamma_i} \right\| \leq \left\| \sum_{i=1}^k a_i v_i \right\| \leq B \left\| \sum_{i=1}^k a_i e_{\gamma_i} \right\| \quad (13)$$

Combining (12) and (13) we finally obtain that for some $D \geq 1$, and any $k \in \mathbb{N}$,

$$\frac{1}{D} \max_j |a_j| \leq \left\| \sum_{i=1}^k a_i e_{\gamma_i} \right\| \leq D \max_j |a_j|. \quad (14)$$

for all $\{\gamma_1, \dots, \gamma_k\} \subset [1, \Lambda)$, which proves our claim. \square

5 Final comments and open problems

Let us mention in this final section some problems of interest.

First of all, we do not know whether or not Theorem A is true if we replace λ by smaller cardinal numbers.

Problem 1. *Assume that X is a Banach space with $\text{dens}(X) \geq \omega_1$, and assume that X coarsely embeds into $c_0(\Gamma)$ for some cardinal number Γ . Does X have trivial co-type? If moreover X has a symmetric basis, must it be isomorphic to $c_0(\omega_1)$?*

Of course Problem 1 would have a positive answer if the following is true.

Problem 2. *Is Theorem B true for ω_1 ?*

Connected to Problems 1 and 2 is the following

Problem 3. Does ℓ_∞ coarsely embed into $c_0(\kappa)$ for some uncountable cardinal number κ .

Another line of interesting problems asks which isomorphic properties do non separable Banach spaces have which coarsely embed into $c_0(\Gamma)$

Problem 4. Does a non separable Banach space which coarsely embeds into some $c_0(\Gamma)$, Γ being uncountable, contain copies c_0 , or even $c_0(\omega_1)$.

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